

A NEW CHARACTERISATION OF GROUPS AMONGST MONOIDS

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ABSTRACT. We prove that a monoid M is a group if and only if, in the category of monoids, all points over M are strong. This sharpens and greatly simplifies a result of Montoli, Rodelo and Van der Linden [8] which characterises groups amongst monoids as the protomodular objects.

In their article [8], Montoli, Rodelo and Van der Linden introduce, amongst other things, the concept of a *protomodular object* in a finitely complete category \mathcal{C} as an object $Y \in \mathcal{C}$ over which all points are *stably strong*. The aim of their definition is two-fold: first of all, to provide a categorical-algebraic characterisation of groups amongst monoids as the protomodular objects in the category \mathbf{Mon} of monoids; and secondly, to establish an object-wise approach to certain important conditions occurring in categorial algebra such as protomodularity [2, 1] and the Mal'tsev axiom [5, 6].

We briefly recall some basic definitions; see [3, 7, 8] for more details. Let \mathcal{C} be a finitely complete category, which we also take to be pointed for the sake of simplicity. In \mathcal{C} , a pair of arrows $(r: W \rightarrow X, s: Y \rightarrow X)$ is **jointly strongly epimorphic** when if $mr' = r$, $ms' = s$ for some given monomorphism $m: M \rightarrow X$ and arrows $r': W \rightarrow M$, $s': Y \rightarrow M$, then m is an isomorphism. In the case of monoids, this means that any $x \in X$ can be written as a product $r(w_1)s(y_1) \cdots r(w_n)s(y_n)$ for some $w_j \in W$, $y_j \in Y$. This characterisation follows easily from the fact that (r, s) is a jointly strongly epimorphic pair in \mathbf{Mon} if and only if the induced monoid morphism $W + Y \rightarrow X$ is a surjection—see, for instance, [1, Corollary A.5.4 combined with Example A.5.16]. Given an object Y in \mathcal{C} , a **point over Y** is a pair of morphisms $(f: X \rightarrow Y, s: Y \rightarrow X)$ such that $fs = 1_Y$. A point (f, s) is said to be **strong** when the pair $(\ker(f): \text{Ker}(f) \rightarrow X, s: Y \rightarrow X)$ is jointly strongly epimorphic. The point (f, s) is **stably strong** when all of its pullbacks are strong. More precisely, if $g: Z \rightarrow Y$ is any morphism, then the pullback $g^*(f)$ together with its splitting induced by s is a strong point.

Even though the concept of a protomodular object serves the intended purpose of characterising groups amongst monoids, the proof of this characterisation given in [8] is rather complicated, since it relies on another, more subtle, characterisation in terms of the so-called *Mal'tsev objects*. The present short note aims to improve the situation by giving a quick and direct proof of a more general result: a monoid is a group as soon as all points over it are strong.

Theorem. *A monoid M is a group if and only if, in \mathbf{Mon} , all points over M are strong.*

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Proof. It is shown in [4]—this is Proposition 2.2.4 combined with Lemma 2.1.6—that for any group M , all points over it are *homogenous*, which makes them (stably) strong. So in particular, if M is a group, then all points over M are strong. We prove the other implication.

Consider $m \in M$ and the induced split extension

$$0 \longrightarrow K \longrightarrow \mathbb{N} + M \begin{array}{c} \xleftarrow{\iota_M} \\ \xrightarrow{(m \ 1_M)} \end{array} M \longrightarrow 0,$$

where $m: \mathbb{N} \rightarrow M$ is the morphism which sends the generator 1 of $(\mathbb{N}, +, 0)$ to the element m of M . By the assumption that $((m \ 1_M), \iota_M)$ is a strong point, $1 \in \mathbb{N}$ can be written as

$$1 = k_1 m_1 \cdots m_i k_{i+1} m_{i+1} \cdots k_n m_n$$

for some $k_j \in K$ and $m_j \in M$. Since 1 is not invertible in \mathbb{N} , it must appear in exactly one of the factors k_j in the product on the right, say in k_{i+1} . Then neither $k_1 m_1 \cdots m_i$ nor $m_{i+1} \cdots k_n m_n$ contains any non-zero elements of \mathbb{N} , so we have that in $\mathbb{N} + M$

$$1 = a' k b'$$

for some $a', b' \in M$ and $k \in K$. Since 1 appears in k we can write $k = a1b$ where $a, b \in M$. Necessarily then $e_M = a'a$ and $e_M = bb'$, because $1 = a'a \cdot 1 \cdot bb'$. Furthermore, since k is in the kernel of $(m \ 1_M)$, we also have that $e_M = amb$. So, clearly, a and b are invertible. As a consequence, m is invertible as well. We conclude that M is a group. \square

Note that the above proof shows in particular why M is **gregarious** in the sense of [1], which means that for any m there exist a and b such that $e_M = amb$. However, the proof also shows that those a and b are invertible, and thus M is a group.

This result seems to indicate that in certain cases (like, for instance, in the category of monoids) it makes sense to weaken the definition of a protomodular object M —all points over M are stably strong—to the condition that those points are strong. This, and related considerations, will be the subject of future joint work with the authors of [8].

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